

Fidelity susceptibility and geometric phase in critical phenomenon*

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Motivated by recent development in quantum fidelity and fidelity susceptibility, we study relations among Lie algebra, fidelity susceptibility and quantum phase transition for two-state system and the Lipkin-Meshkov-Glick model. We get the fidelity susceptibility for $SU(2)$ and $SU(1,1)$ algebraic structure models. From this relation, the validity of the fidelity susceptibility to signal for the quantum phase transition is also verified in these two systems. At the same time, we obtain the geometric phase in these two systems in the process of calculating the fidelity susceptibility. In addition, the new method of calculating fidelity susceptibility has been applied to explore the two-dimensional XXZ model and the Bose-Einstein condensate(BEC).

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1. Introduction

Fidelity, one of the most intriguing feature of quantum information science,^[1] has been widely studied in recent years.^[2–14] The quantum phase transition(QPT), driven purely by quantum fluctuations and occurring at zero temperature, is believed to be an important concept in condensed physics.^[15] In a quantum many-body system, the QPT is driven by purely the quantum fluctuation in ground states. The wave function of the ground state can have a abrupt change as the system varies across the phase transition point. Therefore, an approach to quantum phase transitions based on the quantum-information concept of fidelity has been put forward.^[2] However, except for a few specific models,^[2,16] the calculation of the ground-state

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fidelity is tedious. Recently, a neater and simpler formalism of fidelity to critical phenomena was introduced,^[17] for the so-called fidelity susceptibility, to signal the whole QPT. The main advantage of this approach lies in the fact that the fidelity is a purely Hilbert-space geometrical quantity and no a priori knowledge of the structure of the considered system is required for its use. As shown in Ref. [17], the fidelity susceptibility is intrinsically related to the dynamic structure factor of the driving Hamiltonian that is evaluated through the scheme based on some numerical techniques including exact diagonalization and density matrix renormalization group. On the other hand Zhang *et al.*^[18] employed Lie algebra to evaluate the fidelity susceptibility, and show high efficiency of it.

In this paper, we investigate, in a general framework, how the differential form of the fidelity susceptibility can be established in terms of the general Lie algebra. As will be seen below, under certain conditions, the differential form is an effective tool in detecting the critical points of the QPT. To demonstrate this, analytic formulas for the fidelity and the fidelity susceptibility are derived for the $SU(2)$ and $SU(1,1)$. By using these formulas, the fidelity and the fidelity susceptibility can be easily calculated for a large class of many-body systems, as long as the Hamiltonian of the system can be rewritten as the form [see Eq. (4)]. Employing our general formulas to the two-state system, we can show the fidelity susceptibility of the system in terms of $SU(2)$. At the same time, the geometric phase, which is also an effective indicator in detecting the QPT,^[19–22] can be obtained in the process of calculating the fidelity susceptibility. On the other hand, according to the general expressions of the differential form of the fidelity susceptibility, one can also expect that the same results for the Lipkin-Meshkov-Glick(LMG) model in terms of $SU(1,1)$. Furthermore, we extend this differential form fidelity susceptibility to other physics models.

2. Formulism

The general Hamiltonian of quantum many-body systems reads

$$H(\lambda) = H_0 + \lambda H_I, \quad (1)$$

where H_I is the driving Hamiltonian and λ denotes its strength. The fidelity is the absolute value of the overlap between two ground states $|\Psi_0(\lambda)\rangle$ and $|\Psi_0(\lambda + \delta\lambda)\rangle$,^[2]

$$F(\lambda, \lambda + \delta\lambda) = |\langle \Psi_0(\lambda) | \Psi_0(\lambda + \delta\lambda) \rangle| \quad (2)$$

with $\delta\lambda$ a small deviation. Extracting the leading term of the fidelity, the fidelity susceptibility can be obtained^[17]

$$\chi_F(\lambda) = \sum_{n \neq 0} \frac{|\langle \Psi_n(\lambda) | H_I | \Psi_0(\lambda) \rangle|^2}{[E_n(\lambda) - E_0(\lambda)]^2} \quad (3)$$

with eigenvalues $E_n(\lambda)$ and corresponding normalized eigenvectors $|\Psi_n(\lambda)\rangle$. The eigenstates define a set of orthogonal complete bases in the Hilbert space.

Given a physical system, the Hamiltonian can be written as

$$H = \sum_i \epsilon_i H_i + \sum_\alpha (\lambda_\alpha E_\alpha + \lambda_\alpha^* E_{-\alpha}), \quad (4)$$

where ϵ_i and λ_α are coupling parameters and $\{H_i, E_\alpha, E_{-\alpha}\}$ are the Cartan-Weyl basis of a semi-simple Lie algebra. Using the unitary operator $\hat{U}(\xi_\alpha(\lambda)) = \exp[\sum_\alpha (\xi_\alpha E_\alpha - \xi_\alpha^* E_{-\alpha})]$, the Hamiltonian can be turned into the diagonal form

$$\hat{U}^\dagger(\xi_\alpha) H \hat{U}(\xi_\alpha) = \sum_i \eta_i H_i. \quad (5)$$

Therefore, the eigenstates of Hamiltonian (4) are $|\Psi\rangle = \hat{U}(\xi_\alpha)|ref\rangle$, where $|ref\rangle$ is the direct product of normalized eigenstates of H_i . Then the absolute value of the overlap between $|\Psi(\lambda)\rangle = \hat{U}(\xi_\alpha(\lambda))|ref\rangle$ and $|\Psi(\lambda + \delta\lambda)\rangle = \hat{U}(\xi_\alpha(\lambda + \delta\lambda))|ref\rangle$ is

$$F = |\langle ref | \hat{U}^\dagger(\xi_\alpha(\lambda)) \hat{U}(\xi_\alpha(\lambda + \delta\lambda)) | ref \rangle|. \quad (6)$$

According to the general relation between fidelity and fidelity susceptibility, $F = 1 - \frac{1}{2}(\delta\lambda)^2 \chi(\lambda) + \dots$, the expression of the fidelity susceptibility is given by^[18]

$$\chi(\lambda) = -\langle ref | (\hat{U}^\dagger \partial_\lambda \hat{U})^2 | ref \rangle - |\langle ref | \hat{U}^\dagger \partial_\lambda \hat{U} | ref \rangle|^2. \quad (7)$$

3. Two-state systems

The two-state system is the simplest quantum system which can be calculated exactly. Furthermore, the two-state systems possess several typical quantum properties. So we shall study it firstly. The Hamiltonian of a two-state system can be written as

$$H = -\mathbf{B} \cdot \boldsymbol{\sigma}, \quad (8)$$

where \mathbf{B} is an external magnetic field and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices. In σ_z basis $|\uparrow\rangle, |\downarrow\rangle$, Pauli matrices take the form

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (9)$$

We can rewrite H in the $SU(2)$ form

$$H = -2B \cos \theta J_z - B \sin \theta e^{i\phi} J_+ - B \sin \theta e^{-i\phi} J_-, \quad (10)$$

by means of the generators of the algebra $SU(2)$

$$J_z = \frac{1}{2}\sigma_z, J_+ = \frac{1}{2}\sigma_+, J_- = \frac{1}{2}\sigma_-. \quad (11)$$

These satisfy the usual commutation relations

$$[J_+, J_-] = 2J_z, [J_z, J_\pm] = \pm J_\pm. \quad (12)$$

Resorting to the Eq. (5), we introduce the unitary operator $\hat{U}(\theta, \phi) = \exp(-\frac{\theta}{2}e^{-i\phi}J_+ + \frac{\theta}{2}e^{i\phi}J_-)$.

The Hamiltonian (8) then can be diagonalized

$$\hat{U}^\dagger(\theta, \phi) H \hat{U}(\theta, \phi) = -B J_z. \quad (13)$$

Here, θ and ϕ can be regarded as adiabatic parameters. For simplicity and without loss of generality, we fixed θ first and have

$$\hat{U}^\dagger(\theta, \phi) \partial_\phi \hat{U}(\theta, \phi) = \frac{i}{2} \sin \theta (e^{-i\phi} J_+ + e^{i\phi} J_-) + 2i \sin^2 \frac{\theta}{2} J_z \quad (14)$$

and

$$|\langle ref | \hat{U}^\dagger(\theta, \phi) \partial_\phi \hat{U}(\theta, \phi) | ref \rangle|^2 = 4 \sin^2 \frac{\theta}{4}, \quad (15)$$

where $|ref\rangle$ is the σ_z basis $|\uparrow\rangle, |\downarrow\rangle$ in the two-state system. Finally we get the fidelity susceptibility of spin- $\frac{1}{2}$ subjected to an external magnetic field

$$\chi_F = \frac{1}{4} \sin^2 \theta. \quad (16)$$

Since $\langle ref | \hat{U}^\dagger(\theta, \phi) \partial_\phi \hat{U}(\theta, \phi) | ref \rangle$ is just the Berry adiabatic connection, which contribute a Pancharatnam-Berry phase^[23,24] to the spin as the magnetic field rotates adiabatically around cone direction, we get the Berry phase

$$\begin{aligned} \gamma(\theta, \phi) &= -i \int_0^{2\pi} \langle ref | \hat{U}^\dagger(\theta, \phi) \partial_\phi \hat{U}(\theta, \phi) | ref \rangle \\ &= \pm \pi (1 - \cos \theta). \end{aligned} \quad (17)$$

The results of Eqs. (16) and (17) are in agreement with that in Ref. [25], which verify the reliability of our method.

4. The Lipkin-Meshkov-Glick model

The LMG model^[26–28] was originally introduced in nuclear physics. It provides a simple description of the tunneling of bosons between two degenerate levels and can thus be used to describe many physical systems, such as two-mode Bose-Einstein condensates^[29] and Josephson junctions.^[30] Recently the entanglement in this model has attracted much interest because of available numerical calculations and plentiful phase diagrams.^[31] In the thermodynamic limit, its phase diagram can be simply established by a semiclassical approach.^[32] For finite large number N of particles, it was studied by the $1/N$ expansion in the Holstein-Primakoff single boson representation^[33] and by the continuous unitary transformation.^[34]

The Hamiltonian of the LMG model can be written as

$$\begin{aligned}
H &= -\frac{\lambda}{N} \sum_{i < j} (\sigma_x^i \sigma_x^j + \gamma \sigma_y^i \sigma_y^j + \gamma \sigma_z^i \sigma_z^j) - h \sum_i \sigma_z^i \\
&= -\frac{2\lambda}{N} (S_x^2 + \gamma S_y^2) - 2h S_z + \frac{\lambda}{2} (1 + \gamma) \\
&= -\frac{\lambda}{N} (1 + \gamma) (\mathbf{S}^2 - S_z^2 - \frac{N}{2}) - 2h S_z \\
&\quad - \frac{\lambda}{2N} (1 - \gamma) (S_+^2 + S_-^2),
\end{aligned} \tag{18}$$

where the σ_κ ($\kappa = x, y, z$) are the Pauli matrices, $S_\kappa = \sum_i \sigma_\kappa^i / 2$, and $S_\pm = S_x \pm i S_y$. λ and h are the spin-spin interaction strength and the effective external field, respectively. N is the total spin numbers and $1/N$ ensures that the free energy per spin is finite in the thermodynamical limit. It is understood that H preserve the magnitude of the total spin and the parity $P = \prod_i \sigma_z^i$, i.e.,

$$[H, \mathbf{S}^2] = 0, [H, P] = 0, \tag{19}$$

for all values of the anisotropy parameter γ . Specially, in the isotropic case $\gamma = 1$, one has $[H, S_z] = 0$, so that H is diagonal in the eigenbasis of \mathbf{S}^2 and S_z . We adopt the $1/N$ expansion method corresponding to the large N limit. We first use the Holstein-Primakoff boson representation of the spin operator^[33] in the $S = N/2$ subspace given by

$$\begin{aligned}
S_z &= S - a^\dagger a = \frac{N}{2} - a^\dagger a, \\
S_+ &= (2S - a^\dagger a)^{\frac{1}{2}} a = N^{\frac{1}{2}} (1 - \frac{a^\dagger a}{N})^{\frac{1}{2}} a = S_-^\dagger,
\end{aligned} \tag{20}$$

where the standard bosonic creation and annihilation operator satisfy $[a, a^\dagger] = 1$. This representation is well adapted to the computation of the low-energy physics with $\langle a^\dagger a \rangle / N \ll 1$.

The next step consists in inserting these expression in Eq. (18), and to expand the argument of the square roots. Keeping terms of order $(1/N)^{-1}$, $(1/N)^{-1/2}$, and $(1/N)^0$ in the Hamiltonian yields ($h \geq 1$)

$$H = -hN + (2h - 1 - \gamma)a^\dagger a - [(1 - \gamma)/2](a^{\dagger 2} + a^2). \quad (21)$$

In order to analyze the fidelity susceptibility of the above equation, we transform Eq. (21) into the form of $SU(1, 1)$ following the ideas in Ref. [35]. We first introduce the generators of $SU(1, 1)$,

$$K_+ = \frac{1}{2}a^{\dagger 2}, K_- = \frac{1}{2}a^2, K_z = \frac{1}{4}(2a^\dagger a + 1), \quad (22)$$

which satisfy the communication relations of $SU(1, 1)$ algebra. Submitting these expressions of the $SU(1, 1)$ generators into Eq. (21), one can get

$$H = -hN - \frac{1}{2}(h - \gamma - 1) + 2(2h - \gamma - 1)K_z + (\gamma - 1)K_+ + (\gamma - 1)K_-, \quad (23)$$

which consists with Eq. (4). We do not show the diagonalization of $SU(1, 1)$ algebraic structure model explicitly here, but interested readers are recommended to refer to Ref. [18]. We simply quote the main result that is connection with the fidelity susceptibility of the $SU(1, 1)$ algebra model. After all the procedures, the fidelity susceptibility of $SU(1, 1)$ algebraic structure model becomes

$$\chi_{SU(1,1)} = \frac{1}{8}[(\frac{\partial \theta}{\partial \lambda})^2 + \sinh^2 \theta (\frac{\partial \phi}{\partial \lambda})]. \quad (24)$$

Assuming $\tanh \theta = (1 - \gamma)/(2h - 1 - \gamma)$, the fidelity susceptibility is represented by

$$\chi_F = \frac{(1 - \gamma)^2}{32(1 - h)^2(h - \gamma)^2}, \quad (25)$$

which is in agreement with Ref. [36].

The derivation above is only valid for $h \geq 1$, for $0 < h < 1$ the calculation is actually similar to the above case of $h \geq 1$. When $0 < h < 1$, $\tanh \theta = \frac{h^2 - \gamma}{2 - h^2 - \gamma}$, the fidelity susceptibility is then obtained accordingly

$$\chi_F = \frac{h^2}{8(1 - h^2)^2}. \quad (26)$$

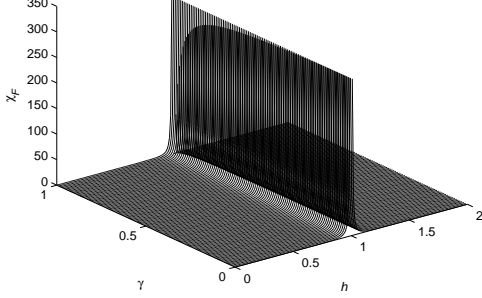


Fig.1. Fidelity susceptibility as a function of h and γ . The divergent character of χ_F is clearly displayed as $h \rightarrow 1$.

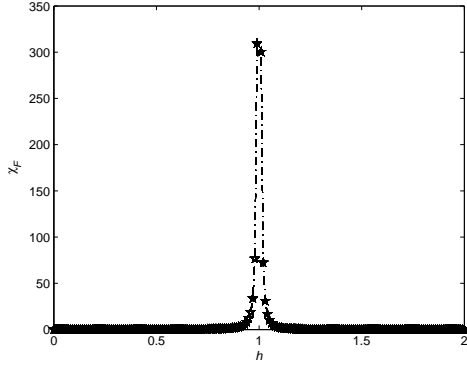


Fig.2. Fidelity susceptibility as a function of h for $\gamma = 0.5$ in large N limit.

Thus we obtained fidelity susceptibility of the anisotropic LMG model in the large N limit. As shown in Figs. 1 and 2, it is obvious that χ_F is divergent at $h = 1$ where the LMG model has been proved to experience a second-order phase transition independent of the anisotropy γ , which is well described by a mean-field approach.^[34] Through the calculation of fidelity susceptibility, we have found the geometric phase. For illustration, we consider the system which has a rotation $g(\phi)$ around the new z direction. The Hamiltonian becomes $H(\phi) = g(\phi)Hg^\dagger(\phi)$ with $g(\phi) = e^{i\phi S_z}$. Then Eq. (21) can be written as

$$H = -hN + (2h - 1 - \gamma)a^\dagger a - [(1 - \gamma)/2](a^{\dagger 2}e^{-2i\phi} + a^2e^{2i\phi}). \quad (27)$$

The geometric phase of the ground state, accumulated by varying the angle ϕ from 0 to π , is described by $\beta = -i \int_0^\pi \langle g | \frac{\partial}{\partial \phi} | g \rangle$.^[19] Finally, we get the geometric phase

$$\begin{aligned} \beta &= -i \int_0^\pi \langle g | \frac{\partial}{\partial \phi} | g \rangle = -i \int_0^\pi \langle 1, 0 | U(\theta, \phi) \partial_\phi U(\theta, \phi) | 1, 0 \rangle \\ &= -i \int_0^\pi -2i \sinh^2 \frac{\theta}{2} = \pi(1 - \cosh \theta). \end{aligned} \quad (28)$$

The derivation above is also only valid for $h \geq 1$. When $0 < h < 1$, $\tanh \theta = \frac{h^2 - \gamma}{2 - h^2 - \gamma}$. Therefore we can get geometric phase of the LMG model in the thermodynamic limit. One can find β_g is also divergent at $h = 1$ which is equal to the fidelity susceptibility in indicating the quantum phase transition in Figs. 3 and 4.

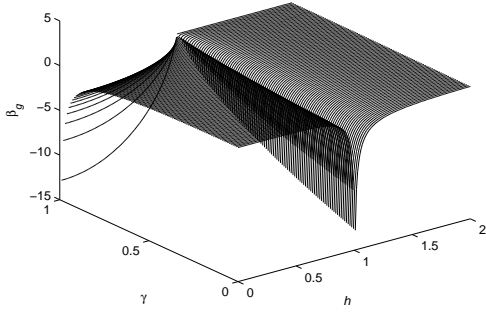


Fig.3. The geometric phase as a function of h and γ . It is obvious that β_g is divergent at $h = 1$ independent of γ .

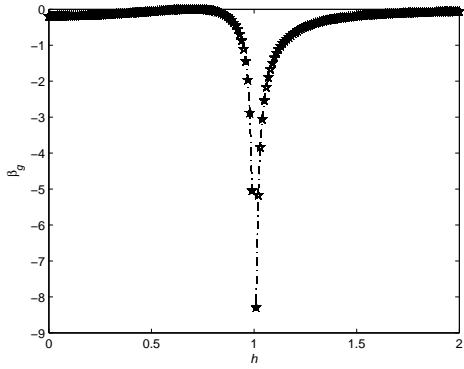


Fig.4. The geometric phase as a function of h . The parameter $\gamma = 0.5$.

5. Other models

In this section, we generalize the new method of calculating fidelity susceptibility to other quantum many-body models. As there is nobody to obtain the analytical results of the fidelity

susceptibility of the two-dimensional XXZ model and the Bose-Einstein condensate precisely, we expect the under results can arouse more wonderful ideas or results.

5.1. Two-dimensional XXZ model

The Hamiltonian of the XXZ antiferromagnetic model reads

$$H_{XXZ} = \sum_{\langle ij \rangle} (S_i^x S_j^x + S_i^y S_j^y + \eta S_i^z S_j^z), \quad (29)$$

where $S_i^\alpha (\alpha = x, y, z)$ are the spin-1/2 operators at site i and $\eta = J_z/J_x (J_x = J_y)$ is a dimensionless parameter characterizing the anisotropy of the model. The sum runs over all the nearest neighbors on a square lattice. We begin with the two-sublattice model and Holstein-Primakoff transformation,^[33]

$$\begin{aligned} S_a^+ &= \sqrt{2S} a^\dagger (1 - \frac{a^\dagger a}{2S})^{1/2}, S_a^- = (S_a^+)^{\dagger}, \\ S_b^+ &= \sqrt{2S} b^\dagger (1 - \frac{b^\dagger b}{2S})^{1/2}, S_b^- = (S_b^+)^{\dagger}, \\ \hat{S}_a^z &= -S + a^\dagger a, \hat{S}_b^z = S - b^\dagger b, \end{aligned} \quad (30)$$

where $a^\dagger, a(b^\dagger, b)$ are boson creation and annihilation operators on sublattice A (sublattice B), respectively. The particle numbers $n_a = a^\dagger a$, $n_b = b^\dagger b$ cannot excel $2S$. Transforming the operators into momentum space, we rewrite the Hamiltonian as

$$H_{XXZ} = -2\eta z N S^2 + 2zS \sum_{\mathbf{k}} H_{\mathbf{k}}, \quad (31)$$

where z is the coordination number of the lattice and $H_{\mathbf{k}}$ is of the form

$$H_{\mathbf{k}} = \eta(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + b_{\mathbf{k}}^\dagger b_{\mathbf{k}}) + \gamma_{\mathbf{k}}(a_{\mathbf{k}} b_{\mathbf{k}} + a_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger). \quad (32)$$

Here $\gamma_{\mathbf{k}} = z^{-1} \sum_{\mathbf{R}} e^{i\mathbf{k} \cdot \mathbf{R}}$ with \mathbf{R} is a vector connecting an atom with its nearest neighbor. We can rewrite $H_{\mathbf{k}}$ in the $SU(1,1)$ form,

$$H_{\mathbf{k}} = 2\eta A_z^{\mathbf{k}} + \gamma_{\mathbf{k}}(A_+^{\mathbf{k}} + A_-^{\mathbf{k}}), \quad (33)$$

by means of the generators of the algebra $SU(1,1)$

$$\begin{aligned} A_+^{\mathbf{k}} &= a_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger, A_-^{\mathbf{k}} = a_{\mathbf{k}} b_{\mathbf{k}}, \\ A_z^{\mathbf{k}} &= \frac{1}{2}(n_{\mathbf{k}}^a + n_{\mathbf{k}}^b + 1). \end{aligned} \quad (34)$$

Taking a similar transformation of $H_{\mathbf{k}}$, we get the unitary operator $U = \exp(\xi_{\mathbf{k}} A_+^{\mathbf{k}} - \xi_{\mathbf{k}} A_-^{\mathbf{k}})$ with $\tan \xi_{\mathbf{k}} = -\frac{\gamma_{\mathbf{k}}}{\eta}$. With the spin-wave theory framework, we obtain the unitary operator and then calculate the fidelity susceptibility of the model in two dimensions. Taking $\tanh \theta = \frac{\gamma_{\mathbf{k}}}{\eta}$, $e^{-i\Phi} = \frac{\gamma_{\mathbf{k}}}{\gamma_{\mathbf{k}}} = 1$ into the Eq. (24), one can obtain

$$\chi = \int \frac{(\eta \partial_{\lambda} \gamma_{\mathbf{k}} - \gamma_{\mathbf{k}} \partial_{\lambda} \eta)^2}{32(\eta^2 - \gamma_{\mathbf{k}}^2)^2} \frac{d\mathbf{k}}{(2\pi)^3}. \quad (35)$$

5.2. Fidelity susceptibility for the Bose-Einstein condensate

The standard description of the Bose-Einstein condensate is via an order parameter field $\Psi(x)$. The Hamiltonian takes the standard form

$$\begin{aligned} \mathcal{H}[\Psi] &= \int d^3x \left[\frac{\hbar^2}{2m} |\nabla \Psi(x)|^2 + U(x) |\Psi(x)|^2 \right] \\ &+ \frac{1}{2} \int d^3x \int d^3y \Psi^*(y) \Psi^*(x) V(x, y) \Psi(y) \Psi(x). \end{aligned} \quad (36)$$

As in Ref. [35], the Eq. (36) can be rewritten in the form of $SU(1, 1)$ algebra

$$\begin{aligned} \mathcal{H} &= 2[\sigma_0 A_3^0 + \frac{1}{2}(u_0 A_+^0 + u_0^* A_-^0)] \\ &+ \sum_{k \neq 0} [\sigma_k A_3^k + \frac{1}{2}(u_k A_+^k + u_k^* A_-^k)] - E_*, \end{aligned} \quad (37)$$

where $\sigma_0 \equiv \epsilon_0 + \frac{1}{2} \sum_{k \neq 0} (V_0 + V_k)(\langle n_k \rangle + \langle n_{-k} \rangle)$, $u_0 \equiv V_0 \langle a_0^2 \rangle + \sum_{k \neq 0} V_k \langle a_k a_{-k} \rangle$, $\sigma_k \equiv \epsilon_k + \langle n_0 \rangle (V_0 + V_k)$, $u_k \equiv V_k \langle a_0^2 \rangle$, $E_* = \frac{1}{2} [V_0 \langle a_0^2 \rangle^2 + \sigma_0] + \frac{1}{2} \sum_{k \neq 0} [(\sigma_k - \epsilon_k) \langle n_k + n_{-k} \rangle + \sigma_k] + \frac{1}{2} \sum_{k \neq 0} (u_k \langle a_k^+ a_{-k}^+ \rangle + u_k^* \langle a_k a_{-k} \rangle)$. Introducing the generators of the algebra $SU(1, 1)$

$$A_3^0 = \frac{1}{2}(n_0 + \frac{1}{2}), A_+^0 = \frac{a_0^{+2}}{2}, A_-^0 = \frac{a_0^2}{2} \quad (38)$$

and

$$A_3^k = \frac{1}{2}(n_k + n_{-k} + 1), A_+^k = a_k^+ a_{-k}^+, A_-^k = a_k a_{-k}. \quad (39)$$

We calculate the fidelity susceptibility of $\mathcal{H}_0 = 2[\sigma_0 A_3^0 + \frac{1}{2}(u_0 A_+^0 + u_0^* A_-^0)]$. From this equation, we can get

$$U = \exp(\xi_0 A_+^0 - \xi_0^* A_-^0), \xi_0 = r \exp(i\phi). \quad (40)$$

Taking the same procedures as in the two-dimensional XXZ model, we can get the fidelity susceptibility of the \mathcal{H}_0

$$\begin{aligned} \chi_0 &= \frac{[\sigma_0(u_0 \partial_{\lambda} u_0^* + u_0^* \partial_{\lambda} \sigma) - 2|u_0|^2 \partial_{\lambda} \sigma]^2}{32|u_0|^2(\sigma_0^2 - |u_0|^2)^2} \\ &- \frac{(\sigma_0^2 - |u_0|^2)(u_0^* \partial_{\lambda} u_0 - u_0 \partial_{\lambda} u_0^*)^2}{32|u_0|^2(\sigma_0^2 - |u_0|^2)^2}. \end{aligned} \quad (41)$$

Similarly the fidelity susceptibility of the \mathcal{H}_k can be obtained

$$\chi_k = \frac{[\sigma_k(u_k \partial_\lambda u_k^* + u_k^* \partial_\lambda \sigma) - 2|u_k|^2 \partial_\lambda \sigma_k]^2}{32|u_k|^2(\sigma_k^2 - |u_k|^2)^2} - \frac{(\sigma_k^2 - |u_k|^2)(u_k^* \partial_\lambda u_k - u_k \partial_\lambda u_k^*)^2}{32|u_k|^2(\sigma_k^2 - |u_k|^2)^2}. \quad (42)$$

6. Conclusions

In conclusion, we have established the differential form of the fidelity susceptibility in terms of the general Lie algebras. Meanwhile we investigate the geometric phase which can also show the phase transition point. Therefore, we construct the relation between the fidelity susceptibility and geometric phase. We also apply the differential form of fidelity susceptibility to other physics models. In particular, we focus on the $SU(2)$ and $SU(1, 1)$ algebras. The form of the fidelity susceptibility of $SU(2)$ and $SU(1, 1)$ algebra not only enables us to evaluate the fidelity susceptibility easily, but also builds a straightforward connection between quantum-information theory and the Lie algebra in quantum many-body physics.

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